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Quantization of kinematics and dynamics on S^1 with difference operators and a related q-deformation of the Witt algebra

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Abstract. Motivated by the fact that momentum observables which are modelled in a classical theory as difference quotients of functions are directly accessible to measurements, whereas differentials are mathematical idealizations, which are obtained from difference quotients via a limiting process, we propose a quantization method in which momentum operators are given in terms of q-derivatives, i.e. a particular type of difference quotient, which is particularly suitable on S^1. The quantization scheme is obtained from Borel quantization, in which momentum operators are given as differential operators, via a q-deformation of the kinematical algebra. It will be applied to a system localized and moving on the N-point discretization S^1_N of S^1 and leads to a discrete, nonlinear Schrödinger equation. In the limit $q \to 1$, i.e. the continuous idealization, we find evolution equations which are special cases of the nonlinear Schrödinger equation derived from Borel quantization on S^1, which is based on the undeformed kinematical algebra. It turns out that with this procedure both the real and imaginary part of the nonlinearity can be derived, which without deformation was only possible for the imaginary part. Hence, one can learn on the situation in the continuous case if it is viewed as a limit of the (q-deformed) case.

1. Introduction

It is suggestive to describe a physical system in terms of those quantities which are measurable. We quote in this connection Heisenberg [14] '... it seems necessary to demand that no concept enters a theory which has not been experimentally verified at least to the same degree of accuracy as the experiments explained by the theory'. The measurement of momentum in classical mechanics in concrete experimental situations is related to two consecutive positional measurements $f(x_1)$ and $f(x_2)$, from which momentum is inferred by the calculation of the corresponding differential quotient

$$\mathrm{D}_* f := \frac{f(x_1) - f(x_2)}{x_1 - x_2}. \tag{1}$$

Hence, it is consequent to model momentum by difference quotients and not by their mathematical idealization, the differentials. This is a known procedure in classical mechanics.

In quantum mechanics, positional observables give the probability to find a particle in a set, call it B, in the (continuous or discrete) configuration space M. The momentum observables are quantized via differential operators and together with the position observables they constitute the kinematics of the system. If one accepts that classical momenta are modelled through difference operators, it is plausible to look for a quantization of momentum in terms of difference operators. A difference operator can be formulated on a configuration space M or a suitable discretization of M as follows: take for example $x \in \mathbb{R}$, $a > 0$, then

$$\mathrm{D}_a\, f(x) = \frac{f(x+a) - f(x-a)}{2a} \tag{2}$$

acts also on the discretization $\mathbb{R}_a = (na, n \in \mathbb{Z})$ which is adapted to the difference operator. Hence, a kinematics on M with difference operators leads also to a kinematics on an appropriate discretization of M.

The introduction of difference operators is not unique and one needs a guiding principle. Here we use the fact that the kinematics of usual quantum mechanics carries a Lie algebra structure (see subsection 2.1), and we want to use such difference operators that the corresponding kinematics arises as a deformation with a reasonable algebraic structure. Here naturally the concept of q-deformation comes in. which means a deformation in the category of Lie algebras, i.e. a deformation of the commutation relations with a parameter $q \in \mathbb{C}$, so that the new algebra allows for realizations in terms of difference quotients. The choice of the difference quotients depends also on the structure of M. In this paper we discuss the simplest compact manifold, i.e. S^1, where multiplicative difference operators, called *q-derivatives*, of the following form arise naturally (the name multiplicative stems from the fact that it is obtained from (1) with $x_1 = qx$ and $x_2 = q^{-1}x$, and q-derivative refers to the fact that the scaling factor is q):

$$\mathrm{D}_q\, f(x) := \frac{f(qx) - f(q^{-1}x)}{(q - q^{-1})x} = \frac{1}{x}[N_x]f(x) \tag{3}$$

where

$$N_x := x\partial_x \tag{4}$$

is known as the homogeneity operator and

$$[a] = [a]_q := \frac{q^a - q^{-a}}{q - q^{-1}} \tag{5}$$

as the q-number of the quantity a, which can be a number or a diagonal operator like N_x. In particular, $[N_x]$ is composed of shift operators $q^{\pm N_x}$ which act on functions $f(x)$ as

$$q^{\pm N_x} f(x) = f(q^{\pm} x). \tag{6}$$

It will be shown in section 4 that for q an Nth root of unity, D_q can be viewed in a natural way as discrete derivative on the N-point discretization S^1_N of S^1, which is given by the Nth roots of unity. We remark that q-derivatives arise in a natural way in the framework of quantum groups. In particular, the q-number notation $[a]$ is also used for diagonal operators already in the defining relations of a quantum group as given by Drinfeld [13].

If the function f depends on two variables, e.g. $f = f(x,t)$, we also use the notation $\mathrm{D}_{q,x}$, or $\mathrm{D}_{q,t}$ respectively in order to specify on which variable the q-derivative acts. A similar derivative has been introduced in [6]. It is defined as

$$\mathrm{D}_J\, f(x) := \frac{f(qx) - f(x)}{(q-1)x} \tag{7}$$

and is related to $[N_z]$ in (3) via

$$[N_z] = \mathrm{D}_J\left(\frac{q^{-N_z}+1}{q^{-1}+1}\right). \tag{8}$$

The q-deformed kinematical algebra, i.e. the classical observables in terms of q-difference operators, are then used to obtain quantum mechanical position and momentum operators in terms of q-derivatives along the lines of Borel quantization (see section 2) and leads to a discrete Schrödinger equation on the N-point discretization S^1_N of S^1. During this process, called *q-Borel quantization*, some physically and mathematically motivated assumptions will be made, which are referred to as *q-assumptions*, because they are related to the introduction of q-derivatives.

As mentioned above, Borel quantization not only covers the kinematics, i.e. position and momentum observables, but also provides a quantum analogue to Newtonian dynamics in terms of an evolution (Schrödinger) equation for the wavefunction. In the undeformed case [10] this evolution equation (Doebner–Goldin equation (DG-equation)) contains a nonlinear complex term, which has been studied recently to some extend (e.g. [8, 11, 18]). It depends on a quantum number D, which is inherent in Borel quantization. However, only the imaginary part of the nonlinearity could be derived explicitly and the real part was fixed by plausible assumptions related to the form of the imaginary part [9].

The discrete Schrödinger equation on S^1_N obtained via q-Borel quantization, denoted as the q-Schrödinger equation because it depends on the deformation parameter $q = \exp(\frac{2\pi \mathrm{i}}{N})$, contains q-difference operators instead of differentials. It displays a peculiarity: it depends on an additional parameter $j \in \mathbb{N}$ which was introduced as a mathematical necessity in connection with the q-deformation of the kinematical algebra (see section 3.1). It introduces an interaction between the points x_j, $j = 1, \ldots, p$ with $x_j = q^j x$ as discussed in section 4.4.

Furthermore, it turns out that the q-evolution on the lattice S^1_N is more restricted than the undeformed DG-equation on S^1, because it not only reproduces the latter in the limit $q \to 1$ (or equivalently $N \to \infty$, because $q = \exp(\frac{2\pi \mathrm{i}}{N})$), but also gives an explicit form for the real part of the nonlinearity, which could not be derived via Borel quantization without deformation of the kinematical algebra.

A q-deformation of quantum Borel kinematics and a related construction of a q-deformed and lattice dynamics from it have not been discussed before. q-Deformations of the Schrödinger algebra [7] and of the Schrödinger equation [2, 17, 20], exist, but were not linked to the above-mentioned quantization procedure. The construction of a quantum Borel kinematics on a discrete version of S^1 was also done in [21], but without the more realistic q-derivatives and without derivation of an evolution equation. For a discussion of quantum mechanics on S^1_N in terms of q-derivatives see also [16].

The paper is organized as follows. In section 2 we introduce the quantum Borel kinematics and recall the results of the quantum Borel kinematics on S^1 [12], giving the position and momentum operators explicitly in terms of the Witt generators and also a family of nonlinear Schrödinger equations. In section 3 we present our q-deformation of the kinematical algebra on S^1 via a q-deformation of the Witt algebra. In section 4 we treat the quantum Borel kinematics on S^1_N, derive a set of nonlinear discrete Schrödinger equations and discuss the limit $q \to 1$. Section 5 contains a summary of the results and an outlook.

2. The quantum Borel kinematics on S^1

2.1. Introducing Borel quantization

Borel quantization is a quantization procedure which is designed for systems localized on a continuous configuration manifold M, especially for M with nontrivial topology. The kinematical part was introduced in [1] and the dynamical part in [8, 10]. We give a short review of this procedure. Consider the classical position and momentum observables, i.e. smooth real functions $f \in C^\infty(M, \mathbb{R})$ and smooth vector fields $X \in \mathrm{Vect}(M)$ on M. These two mathematical objects span an infinite-dimensional Lie algebra

$$\mathcal{S}(M) := C^\infty(M, \mathbb{R}) \oplus\!\!\!\!\!\supset \mathrm{Vect}(M) \tag{9}$$

which is the *natural symmetry algebra* on M, denoted also as invariance or kinematical algebra. Quantization of the kinematical algebra means to construct a (quantization) map from $\mathcal{S}(M)$ into the set of selfadjoint operators in some Hilbert space $\mathcal{H}$

$$\begin{aligned} &C^\infty(M, \mathbb{R}) \ni f \mapsto Q(f) \in SA(\mathcal{H}) \\ &\mathrm{Vect}(M) \ni X \mapsto P(X) \in SA(\mathcal{H}). \end{aligned} \tag{10}$$

With additional, physically motivated assumptions the set of such maps can be obtained and also classified. The classification depends on the topology of M and furthermore on a new quantum number $D \in \mathbb{R}$, which is not related to the topology. These maps are the building blocks of the dynamics of the system moving on M in the following sense: consider a classical point-like system, e.g. in $\mathbb{R}^3$, with Newtonian evolution equations (mass m=1) $\dot{x} = p$, $\dot{p} = F(x, p)$. They imply for the time dependence of a position observable $f(x(t))$ the relation $\dot{f} = (\mathrm{grad}\, f)p$. As indicated in [10] this leads to the following equation between matrix elements (Schrödinger representation) and the corresponding quantized operators given by (10), e.g. for pure states $\psi \in H$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi, Q(f)\psi\rangle = \langle\psi, P(X_{\mathrm{grad}\, f})\psi\rangle \qquad \forall f \in C^\infty(\mathbb{R}^3, \mathbb{R}). \tag{11}$$

This relation may be viewed as a generalization of the first Ehrenfest relation. It restricts the time evolution of pure states ψ and it enforces a nonlinear term in the usual (linear) Schrödinger equation proportional to $\mathrm{i}D$ and an arbitrary nonlinear real part. With some plausible physical assumptions on the real part, a family of nonlinear Schrödinger equations is obtained (Doebner–Goldin family (DG-family) of Schrödinger equations) which contains the usual linear ones for $D = 0$. (For more details see [8, 9, 18].)

2.2. Borel quantization on S^1

We describe how Borel quantization works for S^1, especially because we will tailor the q-deformation along the same lines in section 3. On S^1, we have as position observables $f(\phi)$ and as vector fields $X = X(\phi)\frac{\mathrm{d}}{\mathrm{d}\phi}$, where $f(\phi), X(\phi) \in C^\infty(S^1, \mathbb{R})$. Here $\phi \in [0, 2\pi)$ parametrizes S^1. They span the kinematical algebra $\mathcal{S}(S^1) = C^\infty(S^1, \mathbb{R}) \oplus\!\!\!\!\!\supset \mathrm{Vect}(S^1)$. We represent $\mathcal{S}(S^1)$ by selfadjoint operators in the Hilbert space $L^2(S^1, \mathrm{d}\phi)$ [12]. The commutation relations are

$$\begin{aligned} &[Q(f), Q(g)] = 0 \\ &[P(X), Q(f)] = -\mathrm{i}Q(Xf) \\ &[P(X), P(Y)] = -\mathrm{i}P([X, Y]) \end{aligned} \tag{12}$$

and the representation of $\mathcal{S}(S^1)$ on $C^\infty(S^1, \mathbb{R})$ is given by

$$\begin{aligned}
[Q(f)\psi](\phi) &= f(\phi)\psi(\phi) \\
[P(X)\psi](\phi) &= [-\mathrm{i}X + (-\tfrac{1}{2}\mathrm{i} + D)(\operatorname{div} X) + \omega(X)]\psi(\phi).
\end{aligned} \tag{13}$$

Unitarily inequivalent representations are characterized by the quantum number $D \in \mathbb{R}$ and by a closed one-form ω on S^1, $\omega = \theta \mathrm{d}\phi$ with $\theta \in [0, 1)$, i.e. by two numbers. θ comes in because S^1 is topologically nontrivial. The fact that D is real guarantees that P is selfadjoint.

The operator P is given in terms of the generators of the so-called Witt algebra, and together with Q of the inhomogeneous Witt algebra. To see this, we use a Fourier expansion for $f(\phi)$, $X(\phi) \in C^\infty(S^1, \mathbb{R})$, i.e. with $z := \exp(\mathrm{i}\phi)$

$$\begin{aligned}
f(\phi) &= \sum_{n=-\infty}^{\infty} f_n z^n \\
X(\phi) &= \sum_{n=-\infty}^{\infty} X_n z^n
\end{aligned} \tag{14}$$

in which $f_n = \bar{f}_{-n}$, $X_n = \bar{X}_{-n}$ holds. Then (13) yields:

$$\begin{aligned}
Q(f) &= \sum_{n=-\infty}^{\infty} f_n z^n \\
P(X) &= \sum_{n=-\infty}^{\infty} X_n z^n \left(z \frac{\mathrm{d}}{\mathrm{d}z} + \frac{n}{2} + \theta + \mathrm{i}Dn \right).
\end{aligned} \tag{15}$$

As mentioned already, for each tuple (D, θ), these formulae represent unitarily inequivalent quantizations.

If we introduce the notation

$$\begin{aligned}
T_n &= z^n \\
L_n^\theta &= z^n \left(z \frac{\mathrm{d}}{\mathrm{d}z} + \frac{n}{2} + \theta \right)
\end{aligned} \tag{16}$$

we have

$$\begin{aligned}
Q(f) &= \sum_{n=-\infty}^{\infty} f_n T_n \\
P(X) &= \sum_{n=-\infty}^{\infty} X_n (L_n^\theta + \mathrm{i}DnT_n).
\end{aligned} \tag{17}$$

If $\theta \in \mathbb{R}$ is a *fixed* constant, then the generators T_n and $L_n \equiv L_n^\theta$ fulfil the commutation relations:

$$\begin{aligned}
[T_m, T_n] &= 0 \\
[L_n, T_m] &= mT_{m+n} \\
[L_m, L_n] &= (n - m)L_{m+n}.
\end{aligned} \tag{18}$$

In this case $\{L_n\}$ span the Witt algebra and $\{T_n, L_n\}$ the inhomogeneous Witt algebra, i.e. $\{T_n\} \oplus\!\!\!\!\!\supset \{L_n\}$.

For *variable* θ in L_n^θ we get a more general algebra:

$$\begin{aligned}
&[T_m, T_n] = 0\\
&[L_n^\theta, T_m] = mT_{m+n}\\
&[L_m^{\theta_1}, L_n^{\theta_2}] = \begin{cases} (n-m)L_{m+n}^{\theta_3} & n \neq m, \theta_3 = \frac{n\theta_2 - m\theta_1}{n-m} \\ (\theta_2 - \theta_1)nT_{2n} & n = m \end{cases}.
\end{aligned} \tag{19}$$

Note that the generators L_n^θ for variable θ do not form a closed subalgebra; the corresponding generalized Witt algebra has a more complicated structure than the inhomogeneous Witt algebra which it contains as a subalgebra. The generalized Witt algebra (19) has another subalgebra, namely, the one generated by T_n, L_n^θ with rational $\theta \in \mathbb{Q}$, i.e. $\theta = p/q$, $p, q \in \mathbb{Z}$, $q \neq 0$. Then, for $\theta_i = p_i/q_i$, $i = 1, 2, 3$ in (19) we have $p_3 = nq_1p_2 - mp_1q_2$, $q_3 = q_1q_2(n-m)$. We may call this algebra the *rational Witt algebra*. It also contains the inhomogeneous Witt algebra as a subalgebra.

As indicated earlier, different fixed values of θ correspond to unitarily inequivalent quantizations, and thus the above algebraic structure relates different unitarily inequivalent quantizations in $L^2(S^1, \mathrm{d}\phi)$. The sector for fixed θ is invariant under the inhomogeneous Witt algebra. Analogously, the rational sector with $\theta \in \mathbb{Q}$ is invariant under the rational Witt algebra.

For convenience, we give the commutators (12) between $Q(f)$, $P(X)$ in terms of the L_n^θ and T_n:

$$\begin{aligned}
&[Q(f), Q(g)] = 0\\
&[P(X), Q(f)] = \sum_{n,m} X_n f_m([L_n^\theta, T_m] + \mathrm{i}nD[T_n, T_m]) = \sum_{n,m} X_n f_m [L_n^\theta, T_m] = -\mathrm{i}Q(Xf)\\
&[P(X), P(Y)] = \sum_{n,m} X_n Y_m([L_n^\theta, L_m^\theta] + \mathrm{i}D(m[L_n^\theta, T_m] - n[L_m^\theta, T_n])) = -\mathrm{i}P([X, Y]).
\end{aligned} \tag{20}$$

To define a dynamics on these representations of the kinematical algebra we use the generalized Ehrenfest relation (see (11))

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi(\phi,t), Q(f)\psi(\phi,t)\rangle = \langle\psi(\phi,t), P(X_{\mathrm{grad}\, f})\psi(\phi,t)\rangle \qquad \forall f \in C^\infty(S^1, \mathbb{R}), \psi \in \mathcal{H} \tag{21}$$

where $X_{\mathrm{grad}\, f} = f'(\phi)\frac{\mathrm{d}}{\mathrm{d}\phi}$, or, with the probability density for positions $\rho(\phi,t) = \bar{\psi}(\phi,t)\psi(\phi,t)$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int \rho(\phi,t)f(\phi)\mathrm{d}\phi = &-\int \bar{\psi}(\phi,t)\mathrm{i}f'(\phi,t)\psi'(\phi,t)\mathrm{d}\phi\\
&+\int ((D - \tfrac{1}{2}\mathrm{i})f''(\phi,t)\psi(\phi,t) + \theta f'(\phi,t)\psi(\phi,t))\mathrm{d}\phi
\end{aligned} \tag{22}$$

where $\psi' \equiv \frac{\mathrm{d}\psi}{\mathrm{d}\phi}$. We obtain from (22) after partial integration

$$\int f\dot{\rho} = \mathrm{i}\int f(\bar{\psi}\psi')' + (D - \tfrac{1}{2}\mathrm{i})\int f(\bar{\psi}\psi)'' - \theta\int f(\bar{\psi}\psi)'. \tag{23}$$

Since this has to hold for arbitrary f,

$$\begin{aligned}
\dot{\rho} &= \mathrm{i}(\bar{\psi}\psi')' + (D - \tfrac{1}{2}\mathrm{i})(\bar{\psi}\psi)'' - \theta(\bar{\psi}\psi)'\\
&= \frac{\mathrm{i}}{2}(\bar{\psi}\psi'' - \bar{\psi}''\psi) + D\rho'' - \theta\rho'
\end{aligned}$$

$$= -(j_0^\theta)' + D\rho'' = -(j^\theta)', \; j^\theta = j_0^\theta - D\rho'. \tag{24}$$

Here,

$$j_0^\theta = \frac{\mathrm{i}}{2}(\bar{\psi}'\psi - \bar{\psi}\psi') + \theta\rho \tag{25}$$

is the quantum mechanical current density. Equation (24) is an equation of Fokker–Planck type for $\dot{\rho}$ [8] and restricts the evolution equation for ψ.

With the ansatz (wlog)

$$\begin{aligned} \mathrm{i}\partial_t\psi &= H\psi + G[\bar{\psi},\psi]\psi \\ -\mathrm{i}\partial_t\bar{\psi} &= \bar{H}\bar{\psi} + \bar{G}[\bar{\psi},\psi]\bar{\psi} \end{aligned} \tag{26}$$

where H is a linear operator, which we will later interpret as Hamiltonian, and $G[\bar{\psi},\psi] \equiv \operatorname{Re} G[\bar{\psi},\psi] + \mathrm{i}\operatorname{Im} G[\bar{\psi},\psi]$ a nonlinear function of $\bar{\psi}$, ψ (possibly also on t and ϕ), we obtain

$$\dot{\rho} = \dot{\bar{\psi}}\psi + \bar{\psi}\dot{\psi} = \mathrm{i}(\psi(\bar{H}\bar{\psi}) - \bar{\psi}(H\psi)) + 2\operatorname{Im} G[\bar{\psi},\psi]\rho. \tag{27}$$

This is an information on H and on the imaginary part of G. Together with (24) it gives

$$\operatorname{Im} G[\bar{\psi},\psi] = \frac{D}{2\rho}\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\rho \tag{28}$$

and

$$H\psi = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\psi - \mathrm{i}\theta\frac{\mathrm{d}}{\mathrm{d}\phi}\psi \tag{29}$$

or equivalently

$$H\psi = -\frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}\phi} + \mathrm{i}\theta\right)^2\psi + V\psi \tag{30}$$

with a real potential V, because (27) restricts $H\psi$ only up to a term $c\psi$ with $c \in \mathbb{R}$ a constant. If $D = 0$, the nonlinear term $\operatorname{Im} G[\bar{\psi},\psi]$ vanishes.

The resulting Schrödinger equation is given by ($m = 1$, $\hbar = 1$)

$$\mathrm{i}\partial_t\psi = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\psi - \mathrm{i}\theta\frac{\mathrm{d}}{\mathrm{d}\phi}\psi + \mathrm{i}\frac{D}{2\rho}\left(\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\rho\right)\psi + \underbrace{\operatorname{Re} G[\bar{\psi},\psi]}_{R[\psi]}\psi \tag{31}$$

where the real part $R[\psi]$ of the nonlinear term $G[\bar{\psi},\psi]$ remains undetermined. However, if the following additional assumptions are made, which are motivated by the shape of the imaginary part of $G[\bar{\psi},\psi]$,

(i) $R[\psi]$ should be proportional to D, i.e. vanishing for $D = 0$.

(ii) $R[\psi]$ should have derivatives no higher than of second order and occurring only in the numerator.

(iii) $R[\psi]$ should be complex homogeneous of degree zero, i.e. $R[\alpha\psi] = R[\psi]$ for all $\alpha \in \mathbb{C}$.

Then one gets [8] for the real part the following family

$$R[\psi] := D_1\frac{j_0'}{\rho} + D_2\frac{\rho''}{\rho} + D_3\frac{j_0^2}{\rho^2} + D_4\frac{(j_0\rho')}{\rho^2} + D_5\frac{(\rho')^2}{\rho^2}. \tag{32}$$

Together with (31) it yields a family of nonlinear Schrödinger equations (DG-family) which depends on five additional real parameters. In this way, we get via Borel quantization with (20) and (31) a quantum mechanical description of a system localized on S^1. Its physical importance lies in the fact that it allows for the description of dissipation and of nonlinear effects not only on S^1 but analogously, e.g. on $\mathbb{R}^n$. We emphasize, that the q-deformation of (31) which is derived in the following opens new possibilities which lead to further effects in q-deformed quantum mechanics.

3. q-Deformation of the representations of the kinematical algebra on S^1

In this section we construct a q-deformation of quantum Borel kinematics on S^1. We present plausible arguments and assumptions, denoted as *q-assumptions*, to replace all derivatives by more realistic q-derivatives. In the limit $q \to 1$ we expect the results obtained in section 2.2. We parametrize functions f on S^1 as in section 2 by $z = \exp(\mathrm{i}\phi)$. This implies $\partial_\phi f(\phi) = \mathrm{i}N_z f(z)$ and a q-analogue is

$$\mathrm{i}[N_z] f(z) = \frac{i}{q - q^{-1}} (f(qz) - f(q^{-1}z)) \tag{33}$$

with $\lim_{q\to 1}[N_z] = N_z$. It will arise naturally via q-deformation of the kinematical algebra as presented below.

3.1. The q-Witt algebra and the inhomogeneous q-Witt algebra

The kinematical algebra on S^1 is the inhomogeneous Witt algebra (18) in 2. To introduce q-derivatives, we construct a q-deformation of L_n, i.e. a q-Witt algebra, which reproduces (19) in the limit $q \to 1$. There are results on the q-deformation of the Witt algebra [3, 15, 19], but we need a version which is adapted to our purposes, i.e. we make the *q-assumption* that the deformation should introduce as little extra parameters as possible and no extra derivatives into the generators. The following 1-parameter set with parameter $j \in \mathbb{R}_+$ or $j \in \mathbb{N}$ will meet these assumptions†:

$$\mathcal{L}_m^{(j,\theta)} := z^m \frac{\left[j\left(N_z + \frac{m}{2} + \theta\right)\right]}{[j]}. \tag{34}$$

The extra parameter j is needed to close the q-deformed Witt generators $\mathcal{L}_m^{(j,\theta)}$ to a Lie algebra together with the generators T_n which remain undeformed:

$$[\mathcal{L}_m^{(j_1,\theta_1)}, \mathcal{L}_n^{(j_2,\theta_2)}] = \frac{\left[j_1\frac{n}{2} - j_2\frac{m}{2}\right][j_1 + j_2]}{[j_1][j_2]}\mathcal{L}_{m+n}^{(j_1+j_2,\theta_3)} + \frac{\left[j_1\frac{n}{2} + j_2\frac{m}{2}\right][j_2 - j_1]}{[j_1][j_2]}\mathcal{L}_{m+n}^{(j_2-j_1,\theta_4)} \tag{35}$$

with

$$\begin{aligned} \theta_3 &= \frac{j_1\theta_1 + j_2\theta_2}{j_1 + j_2} \\ \theta_4 &= \frac{j_1\theta_1 - j_2\theta_2}{j_1 - j_2} \end{aligned} \tag{36}$$

for $j_1 \neq j_2$ and

$$[\mathcal{L}_m^{(j,\theta_1)}, \mathcal{L}_n^{(j,\theta_2)}] = \frac{[j(n-m)][2j]}{[j]^2}\mathcal{L}_{m+n}^{\left(2j,\frac{\theta_1+\theta_2}{2}\right)} + \frac{\left[j\frac{n+m}{2}\right][j(\theta_2-\theta_1)]}{[j]^2}T_{m+n} \tag{37}$$

otherwise. For $q = 1$ we get (19). We note that our q-Witt algebra closes with respect to the usual commutator and has a trivial Hopf algebra structure.

† We note that an ansatz with $\left[j\left(N_z + \frac{m}{2} + \theta + \mathrm{i}Dm\right)\right]$ in (15) is possible only if D is imaginary, which, however, would lead to a $P(X)$ which is not self-adjoint.

Remark. In [15] another similar q-deformation of the Witt algebra has been presented. The generators were given by

$$\mathcal{L}_m^{(j,r,\pm)} = z^m \frac{[j(N_z + \frac{m}{2})]_\mp}{[j]_\mp} \frac{[rN_z]_\pm}{[r]_\pm} \tag{38}$$

with two different q-number notations $[n]_- = [n]$ and

$$[n]_+ = \frac{q^n + q^{-n}}{2} \tag{39}$$

and two parameters $j, r \in \mathbb{R}$. This choice of a q-Witt algebra, however, is not useful here, because of our q-assumptions.

Concerning the inhomogeneous Witt algebra related to (34) we remark that the first relation in (18) is not changed, because T_n remains undeformed. Instead of the second relation in (18) we have using (34):

$$\mathcal{L}_m^{(j,\theta)} T_n = T_n \mathcal{L}_m^{(j,\theta+n)} \tag{40}$$

which can be expressed as a commutator as follows:

$$[\mathcal{L}_m^{(j,\theta)}, T_n] = T_n(\mathcal{L}_m^{(j,\theta+n)} - \mathcal{L}_m^{(j,\theta)}) \tag{41}$$

So we get for a q-deformation of $\{T_n^\theta\} \oplus\!\!\!\!\supset \{L_n^\theta\}$ an infinite dimensional vector space spanned by $\mathcal{L}_m^{(j,\theta)}$ and T_n which is closed as a quadratic algebra. Hence, a deformation of subalgebras A and B in a semidirect sum $A \oplus\!\!\!\!\supset B$ yields a deformation of $A \oplus\!\!\!\!\supset B$ only if the deformation of the semidirect sum is adequately defined, in our example with a quadratic commutator.

3.2. *q-Deformation of quantum Borel kinematics on S^1*

We connect the q-deformed $\{T_n^\theta\} \oplus\!\!\!\!\supset \{\mathcal{L}_n^{(j,\theta)}\}$-algebra with a q-deformation of the quantum Borel kinematics (17). Since the subalgebra $\{T_n\}$ is unchanged, we obtain for $Q(f)$

$$Q_q(f) = \sum_{n=-\infty}^{\infty} f_n T_n (= Q(f)). \tag{42}$$

For $P(X)$ the situation is complicated because it has projection parts in $\{T_n\}$ as well as $\{L_n^\theta\}$. The projection part in $\{L_n^\theta\}$ is

$$P(X)|_{\{L_n^\theta\}} = \sum_{n=-\infty}^{\infty} X_n L_n^\theta. \tag{43}$$

A q-deformation according to our deformation rule yields

$$P_q^j(X)|_{\{\mathcal{L}_n^{(j,\theta)}\}} = \sum_{n=-\infty}^{\infty} X_n \mathcal{L}_n^{(j,\theta)}. \tag{44}$$

For the projection part in $\{T_n\}$,

$$P(X)|_{\{T_n\}} = \mathrm{i} \sum_{n=-\infty}^{\infty} X_n n D T_n \tag{45}$$

which formally contains a derivative

$$P(X)|_{\{T_n\}} = \mathrm{i}(N_z) \sum_{n=-\infty}^{\infty} X_n D T_n \tag{46}$$

it is consequent to apply the deformation rule also here. This is again a *q-assumption*. We get

$$P_q^j(X)|_{\{T_n\}} = \mathrm{i}[N_z] \sum_{n=-\infty}^{\infty} X_n DT_n = \mathrm{i} \sum_{n=-\infty}^{\infty} X_n [n] DT_n \tag{47}$$

i.e. the coefficient n is replaced by the q-number $[n]$. So we have for the *q-quantum Borel kinematics on* S^1

$$\begin{aligned} Q_q(f) &= \sum_{n=-\infty}^{\infty} f_n T_n (= Q(f)) \\ P_q^j(X) &= \sum_{n=-\infty}^{\infty} X_n (\mathcal{L}_n^{(j,\theta)} + \mathrm{i}[n] DT_n). \end{aligned} \tag{48}$$

These deformed subalgebras can be combined in a semidirect sum and yield the following commutation relations (note that θ is fixed as in (20)):

$$[Q_q(f), Q_q(g)] = 0 \tag{49}$$

$$\begin{aligned} [P_q(X), Q_q(f)] &:= \sum_{n,m} X_n f_m ([\mathcal{L}_n^{(j,\theta)}, T_m] + \mathrm{i}[n] D[T_n, T_m]) \\ &= \sum_{n,m} X_n f_m [\mathcal{L}_n^{(j,\theta)}, T_m] \end{aligned} \tag{50}$$

$$[P_q(X), P_q(Y)] = \sum_{n,m} X_n Y_m ([\mathcal{L}_n^{(j,\theta)}, \mathcal{L}_m^{(j,\theta)}].{+}\mathrm{i}D([m][\mathcal{L}_n^{(j,\theta)}, T_m] - [n][\mathcal{L}_m^{(j,\theta)}, T_n])). \tag{51}$$

Note that if we use n instead of $[n]$ the corresponding relation for (51) would also hold. However, in this case, the dynamics on S_N^1 derived later (section 4) would be given in terms of two types of derivatives.

4. q-Deformation of representations of the Borel quantization on S_N^1

4.1. q-Deformation of quantum Borel kinematics on S_N^1

A restriction of the difference operator $\mathrm{i}[N_z]$, which due to (33) are appropriate discrete analogues for the differential ∂_ϕ, onto an N-point discretization S_N^1 of S^1, is only possible for particular values of the deformation parameter q. To see this, parametrize the equidistant points in S_N^1 with $l = 0, \ldots, N-1$. The wavefunctions ψ span a finite dimensional Hilbert space $\mathcal{H}_N$ of sequences $\psi = (\psi(0), \ldots, \psi(N-1))$. With a discrete Fourier transform we have

$$\psi(l) = \sum_{n=0}^{N-1} \psi_{nl} z_l^n \text{with } z_l^n = \exp\left(\frac{2\pi\mathrm{i}}{N} nl\right). \tag{52}$$

The action of $\mathrm{i}[N_z]$ on such $\psi(l)$ is $\mathrm{i}[N_z]\psi(l) = \frac{\mathrm{i}}{q-q^{-1}} \sum_n \psi_{n,l}((qz_l)^n - (q^{-1}z_l)^n)$. It is only well defined on S_N^1 if qz_l and $q^{-1}z_l$ are again lattice points. Together with the definition of $z_l = z_l^1$ in (52), this *enforces* for q a value

$$q = \exp\left(\mathrm{i}\frac{2\pi}{N}\right) \tag{53}$$

i.e. q has to be an Nth root of unity. We remark that in the representation theory of simple quantum algebras a choice $q^N = 1$ is a special case leading to an extra structure, cf e.g.

[4, 5]. Here, q with $q^N = 1$ is the natural choice. As a consequence, the limit $q \to 1$ coincides with the limit $N \to \infty$.

The scalar product in $\mathcal{H}_N$ is

$$\langle \psi, \phi \rangle_d := \frac{1}{N} \sum_l \bar{\psi}(l)\phi(l) \tag{54}$$

with

$$\langle z_l^k, z_l^m \rangle_d = \frac{1}{N} \sum_l \bar{z}_l^k z_l^m = \delta_{km}. \tag{55}$$

For convenience we skip the index l at the z-coordinates and write $\psi(z)$ or $\psi(l)$ for $\psi(z_l)$. On S_N^1, the inhomogeneous Witt algebra is finitely generated and one finds for the symmetry algebra $Q_q(f)$ and $P_q(X)$ on $\mathcal{H}_N$ (the index d means discrete)

$$\begin{aligned} Q_q^d(f) &= \sum_{n=0}^{N-1} f_n T_n \qquad (= Q(f)) \\ P_q^{j,d}(X) &= \sum_{n=0}^{N-1} X_n(\mathcal{L}_n^{(j,\theta)} + \mathrm{i}[n]DT_n). \end{aligned} \tag{56}$$

j has to be restricted to $\mathbb{N}$ in order to have an action of $\mathcal{L}_n^{(j,\theta)}$ on $\mathcal{H}_N$. This is due to the fact that $\mathcal{L}_n^{(j,\theta)}$ contains shift-operators of the form q^{jN_z}, which applied to $\psi \in \mathcal{H}_N$ give $q^{jN_z}\psi(l) = \sum_n \psi_{n,l}(q^j z_l)^n$. But for z_l as defined in (52) and q given by (53) $q^j z_l$ is only a lattice point if the above restriction for j is made. The parameter j in $\mathcal{L}_n^{(j,\theta)}$ leads to the fact that $\mathcal{L}_n^{(j,\theta)}$ contains—dependent on j—differences between different points of S_N^1, e.g. between next-nearest neighbours for $j = 2$ or even further points and not only nearest neighbours ($j = 1$). This effect is called topological interaction and will be discussed in section 4.4 in connection with the dynamics. We recall that the introduction of j was mathematically necessary in order to close the q-Witt algebra so that this interaction effect arises naturally. Finally we remark that the q-quantum Borel kinematics in (56) is self-adjoint on $\mathcal{H}_N$.

4.2. q-Deformation of the dynamics on S_N^1

The generalized Ehrenfest relation on $\mathcal{H}_N$ (compare with (21)) reads

$$\mathrm{D}_t\langle \psi(t,z), Q_q^d(f)\psi(t,z)\rangle_d = \langle \psi(t,z), P_q^{j,d}(X_{\mathrm{grad}_q f})\psi(t,z)\rangle_d \tag{57}$$

where D_t denotes the time derivative. In this section we use $D_t \equiv \partial_t$ throughout; a possible replacement by a q-derivative is discussed in section 4.3. The operators $Q_q^d(f)$ and $P_q^{j,d}(X_{\mathrm{grad}_q f})$ are given by (56) and the vector field $X_{\mathrm{grad}_q f}$ is derived from $X_{\mathrm{grad}\, f}$ via a replacement of the derivative by a q-derivative (again a *q-assumption*), i.e. in

$$\begin{aligned} f(z) &= \sum_{n=0}^{N-1} f_n z^n \\ X_{\mathrm{grad}\, f} &= \sum_{n=0}^{N-1} \mathrm{i}n f_n z^n(\mathrm{i}N_z) \end{aligned} \tag{58}$$

we replace n by $[n]$ for the same reasons as in section 3.2. We find with (56)

$$P_q^{j,d}(X_{\mathrm{grad}_q f}) = \sum_{n=0}^{N-1} \mathrm{i}[n] f_n(\mathcal{L}_n^{(j,\theta)} + \mathrm{i}[n]DT_n). \tag{59}$$

We note, that for $q = 1$ (57) reduces to (21) as expected. For the right-hand side of (57) one obtains with (59)

$$\begin{aligned}\langle\psi, P_q^{j,d}((X_{\mathrm{grad}_q} f)\psi\rangle_d &= \frac{1}{N}\sum_{l,k,n,m}\bar{\psi}_k z^{-k}\left(\mathrm{i}[n] f_n \mathcal{L}_n^{(j,\theta)} z^m - D[n]^2 f_n z^n z^m\right)\\ &= \frac{1}{N}\sum_{l,k,n,m}\bar{\psi}_k z^{-k}\psi_m z^m f_n z^n \mathrm{i}[(k-m)]\left(\frac{\left[j\left(m+\frac{k-m}{2}+\theta\right)\right]}{[j]} + \mathrm{i}D[(k-m)]\right)\\ &= \left\langle f, \sum_{k,m}\bar{\psi}_k z^{-k}\psi_m z^m i[(k-m)]\left(\frac{\left[j\left(m+\frac{k-m}{2}+\theta\right)\right]}{[j]} + \mathrm{i}D[(k-m)]\right)\right\rangle_d.\end{aligned} \tag{60}$$

Due to (55), all terms with $k \neq n+m$ are zero and it was thus possible to replace n by $k-m$.

Using that the left-hand side of (57) can be written as

$$\mathrm{D}_t\langle\psi(t,z), Q_q^d(f)\psi(t,z)\rangle_d = \langle f, \mathrm{D}_t\,\rho\rangle_d \tag{61}$$

because the functions f do not depend on t (compare with the undeformed case in section 2.2), we obtain

$$\partial_t\rho = \sum_{k,m}\bar{\psi}_k z^{-k}\psi_m z^m \mathrm{i}[(k-m)]\left(\frac{\left[j\left(m+\frac{k-m}{2}+\theta\right)\right]}{[j]} + \mathrm{i}D[(k-m)]\right). \tag{62}$$

For the first term in the bracket on the right-hand side, we use q-number calculus, i.e. for $a, b \in \mathbb{C}$, $\varepsilon \in \{\pm 1\}$,

$$[a+b] = [a]q^{\varepsilon b} + [b]q^{-\varepsilon a} \tag{63}$$

in which ε opens the possibility for a later appropriate choice. We obtain

$$\sum_{k,m}\bar{\psi}_k z^{-k}\psi_m z^m \mathrm{i}[(k-m)]\frac{\left[j\left(m+\frac{k-m}{2}+\theta\right)\right]}{[j]} = \sum_{k,m}\bar{\psi}_k z^{-k}\psi_m z^m \mathrm{i}\frac{1}{[j]}A(k,m,j,\theta) \tag{64}$$

where

$$\begin{aligned}A(k,m,j,\theta) &= [k]\left[j\frac{(k+\theta)}{2}\right]q^{-\varepsilon_1 m+\varepsilon_2 j\frac{(m+\theta)}{2}} - [m]\left[j\frac{(m+\theta)}{2}\right]q^{-\varepsilon_1 k-\varepsilon_2 j\frac{(k+\theta)}{2}}\\ &\quad+[k]\left[j\frac{(m+\theta)}{2}\right]q^{-\varepsilon_1 m-\varepsilon_2 j\frac{(k+\theta)}{2}} - [m]\left[j\frac{(k+\theta)}{2}\right]q^{-\varepsilon_1 k+\varepsilon_2 j\frac{(m+\theta)}{2}}.\end{aligned} \tag{65}$$

With this, (62) can be written (recall (52)) as

$$\begin{aligned}\partial_t\rho = \frac{\mathrm{i}}{[j]}\Bigg\{&\left([N_z]\left[j\frac{(N_z-\theta)}{2}\right]\bar{\psi}\right)(q^{-\varepsilon_1 N_z+\varepsilon_2 j\frac{(N_z+\theta)}{2}}\psi)\\ &-\left([N_z]\left[j\frac{(N_z+\theta)}{2}\right]\psi\right)(q^{\varepsilon_1 N_z+\varepsilon_2 j\frac{(N_z-\theta)}{2}}\bar{\psi})\\ &-\left([N_z]\left[j\frac{\theta}{2}\right]q^{\varepsilon_2 j\frac{(N_z-\theta)}{2}}\bar{\psi}\right)(q^{-\varepsilon_3\frac{j}{2}N_z-\varepsilon_1 N_z}\psi)\\ &-\left([N_z]\left[j\frac{\theta}{2}\right]q^{\varepsilon_2 j\frac{(N_z+\theta)}{2}}\psi\right)(q^{\varepsilon_4\frac{j}{2}N_z+\varepsilon_1 N_z}\bar{\psi})\\ &-([N_z]q^{\varepsilon_2 j\frac{(N_z-\theta)}{2}}\bar{\psi})\left(\left[j\frac{N_z}{2}\right]q^{-\varepsilon_1 N_z}\psi\right)q^{\varepsilon_3 j\frac{\theta}{2}}\\ &+([N_z]q^{\varepsilon_2 j\frac{(N_z+\theta)}{2}}\psi)\left(\left[j\frac{N_z}{2}\right]q^{\varepsilon_1 N_z}\bar{\psi}\right)q^{\varepsilon_4 j\frac{\theta}{2}}\Bigg\} - D([N_z]^2\bar{\psi}\psi)\end{aligned} \tag{66}$$

where q^{aN_z} acts as a shift-operator on functions $f(z)$, i.e.

$$q^{aN_z} f(z) = f(q^a z). \tag{67}$$

With the choice $\varepsilon_3 = -\varepsilon_4 = \varepsilon_2$ in (66) $\partial_t \rho$ is real. We introduce the notation

$$\partial_t \rho \equiv B(\varepsilon_1, \varepsilon_2, j) - D([N_z]^2 \bar{\psi} \psi). \tag{68}$$

Since $[j\frac{N_z}{2}]$ is composed of shifts $q^{\frac{j}{2}N_z}$, it is only well defined in our discrete setting—for the same reasons as explained under (56)—if we restrict $j \in \mathbb{N}$ further to $j \in 2\mathbb{N}$ (a q-*assumption*).

It is interesting to calculate at this stage the q-analogue j_q^θ of the quantum mechanical current density (25) by means of

$$-\mathrm{i}[N_z] j_q^\theta = \partial_t \rho \qquad \text{for } D = 0. \tag{69}$$

We obtain:

$$j_0^{(q,\theta)} = \frac{1}{[j]} \left(\left[j \frac{(N_z + \theta)}{2} \right] \psi \right) \left(q^{\varepsilon_2 j \frac{(N_z - \theta)}{2}} \bar{\psi} \right) - \frac{1}{[j]} \left(\left[j \frac{(N_z - \theta)}{2} \right] \bar{\psi} \right) \left(q^{\varepsilon_2 j \frac{(N_z + \theta)}{2}} \psi \right). \tag{70}$$

In the limit $q \to 1$, i.e. $N \to \infty$, it reproduces the current (25) on 'S^1_∞' $= S^1$.

The D-independent term $B(\varepsilon_1, \varepsilon_2, j)$ of (66) can be simplified, if we decompose $\psi = \psi_1 + \mathrm{i}\psi_2$:

$$\begin{aligned}
B(\varepsilon_1, \varepsilon_2, j) = \frac{1}{[j]} \Bigg\{ & \left(-[N_z] \left[j \frac{N_z}{2} \right] \psi_1 \right) ((q^{\varepsilon_1 N_z} + q^{-\varepsilon_1 N_z}) q^{\varepsilon_2 j \frac{N_z}{2}} \psi_2) \\
& + \left([N_z] \left[j \frac{N_z}{2} \right] \psi_2 \right) ((q^{\varepsilon_1 N_z} + q^{-\varepsilon_1 N_z}) q^{\varepsilon_2 j \frac{N_z}{2}} \psi_1) \\
& + (-[N_z] q^{\varepsilon_2 \frac{j}{2} N_z} \psi_2) \left(\left[j \frac{N_z}{2} \right] (q^{\varepsilon_1 N_z} + q^{-\varepsilon_1 N_z}) \psi_1 \right) \\
& + ([N_z] q^{\varepsilon_2 \frac{j}{2} N_z} \psi_1) \left(\left[j \frac{N_z}{2} \right] (q^{\varepsilon_1 N_z} + q^{-\varepsilon_1 N_z}) \psi_2 \right) \Bigg\} \\
& + \cos\left(\frac{2\pi}{N} \varepsilon_2 \right) \left[j \frac{\theta}{2} \right] \Bigg\{ (-[N_z] q^{-\varepsilon_2 j \frac{N_z}{2}} \psi_1)((q^{\varepsilon_1 N_z} + q^{-\varepsilon_1 N_z}) q^{\varepsilon_2 j \frac{N_z}{2}} \psi_1) \qquad (71) \\
& \qquad + (-[N_z] q^{-\varepsilon_2 j \frac{N_z}{2}} \psi_2)((q^{\varepsilon_1 N_z} + q^{-\varepsilon_1 N_z}) q^{\varepsilon_2 j \frac{N_z}{2}} \psi_2) \\
& \qquad + (-[N_z] q^{\varepsilon_2 j \frac{N_z}{2}} \psi_1)((q^{\varepsilon_1 N_z} + q^{-\varepsilon_1 N_z}) q^{-\varepsilon_2 j \frac{N_z}{2}} \psi_1) \\
& \qquad + (-[N_z] q^{\varepsilon_2 j \frac{N_z}{2}} \psi_2)((q^{\varepsilon_1 N_z} + q^{-\varepsilon_1 N_z}) q^{-\varepsilon_2 j \frac{N_z}{2}} \psi_2) \Bigg\} \\
& + \sin\left(\frac{2\pi}{N} \varepsilon_2 \right) \left[j \frac{\theta}{2} \right] \Bigg\{ ([N_z] q^{-\varepsilon_2 j \frac{N_z}{2}} \psi_1)((q^{\varepsilon_1 N_z} + q^{-\varepsilon_1 N_z}) q^{\varepsilon_2 j \frac{N_z}{2}} \psi_2) \\
& \qquad + (-[N_z] q^{-\varepsilon_2 j \frac{N_z}{2}} \psi_2)((q^{\varepsilon_1 N_z} + q^{-\varepsilon_1 N_z}) q^{\varepsilon_2 j \frac{N_z}{2}} \psi_1) \\
& \qquad + ([N_z] q^{\varepsilon_2 j \frac{N_z}{2}} \psi_2)((q^{\varepsilon_1 N_z} + q^{-\varepsilon_1 N_z}) q^{-\varepsilon_2 j \frac{N_z}{2}} \psi_1) \\
& \qquad + (-[N_z] q^{\varepsilon_2 j \frac{N_z}{2}} \psi_1)((q^{\varepsilon_1 N_z} + q^{-\varepsilon_1 N_z}) q^{-\varepsilon_2 j \frac{N_z}{2}} \psi_2) \Bigg\}.
\end{aligned} \tag{72}$$

The third and fourth line in (71) are nonlinear, all other terms are linear. The nonlinear terms are independent of θ and come in addition to the nonlinearity proportional to D which arises from the D-dependent term in (68). We assume in the following $\theta = 0$ and indicate the result for $\theta \neq 0$.

Similarly as (27) in the undeformed case, equation (66) is a constraint for the evolution equation for ψ, i.e. for $\mathrm{i}\partial_t\psi$, and in analogy to the construction in section 2.2, the following ansatz is plausible (compare with (26) and (27)) with H_q^j linear in ψ and $G_q^j[\bar{\psi}, \psi]$ nonlinear in ψ, $\bar{\psi}$,

$$\begin{aligned} \mathrm{i}\partial_t\psi\bar{\psi} &= (H_q^j\psi)(S\bar{\psi}) + (G_q^j[\bar{\psi}, \psi]\psi)(R\bar{\psi}) \\ -\mathrm{i}\partial_t\bar{\psi}\psi &= (\bar{H}_q^j\bar{\psi})(\bar{S}\psi) + (\bar{G}_q^j[\bar{\psi}, \psi]\bar{\psi})(\bar{R}\psi). \end{aligned} \tag{73}$$

Here S and R are shift operators as in (67), which typically occur in a q-deformed theory. Equation (73) reduces to (26) in the limit $q \to 1$, if H_q^j and $G_q^j[\bar{\psi}, \psi]$ are such that they give H and $G[\bar{\psi}, \psi]$ in this limit. Inserting shift operators in (73) is again a *q-assumption*. The resulting q-Schrödinger equation is nonlinear because of $S\bar{\psi}$ and $\bar{S}\psi$ and because of the nonlinear function $G_q^j[\bar{\psi}, \psi]$ of ψ and $\bar{\psi}$, i.e. the nonlinearity comes from two sources. The source coming from the shifts S and R vanishes for $q \to 1$—the nonlinear term $G_q^j[\bar{\psi}, \psi]$ remains.

With $H_q^j = H_1 + \mathrm{i}H_2$, $G_q^j[\bar{\psi}, \psi] = G_1 + \mathrm{i}G_2$ (we skip the index j in the following), $S = S_1 + \mathrm{i}S_2$, $R = R_1 + \mathrm{i}R_2$, ψ as above and with (73) we obtain

$$\begin{aligned} \partial_t\rho &= (\partial_t\bar{\psi})\psi + (\partial_t\psi)\bar{\psi} \\ &= \mathrm{i}\{(\bar{H}_q^j\bar{\psi})(\bar{S}\psi) + (\bar{G}_q^j\bar{\psi})(R\psi) - (H_q^j\psi)(S\bar{\psi}) - (G_q^j\psi)(R\bar{\psi})\} \\ &= 2\{(H_1\psi_1)(S_2\psi_1 - S_1\psi_2) + (H_2\psi_2)(S_1\psi_2 - S_2\psi_1) \\ &\quad + (H_2\psi_1)(S_1\psi_1 + S_2\psi_2) + (H_1\psi_2)(S_1\psi_1 + S_2\psi_2)\} \\ &\quad + 2\{(G_1\psi_1)(R_2\psi_1 - R_1\psi_2) + (G_2\psi_2)(R_1\psi_2 - R_2\psi_1) \\ &\quad + (G_2\psi_1)(R_1\psi_1 + R_2\psi_2) + (G_1\psi_2)(R_1\psi_1 + R_2\psi_2)\}. \end{aligned} \tag{74}$$

This has to be equated with (68), where $B(\varepsilon_1, \varepsilon_2, j)$ is given by (71). Together with (74) we conclude for the linear part that

$$\begin{aligned} H_1 &= [N_z]\left[j\frac{N_z}{2}\right] \\ H_2 &= 0 \\ S_1 &= \tfrac{1}{2}(q^{\varepsilon_1 N_z} + q^{-\varepsilon_1 N_z})q^{\varepsilon_2 j\frac{N_z}{2}} \\ S_2 &= 0. \end{aligned} \tag{75}$$

The result is ($j \in 2\mathbb{N}$, ε_1, $\varepsilon_2 = \pm 1$)

$$(H_q^j\psi)(S\bar{\psi}) = \left([N_z]\left[j\frac{N_z}{2}\right][j]^{-1}\psi\right)\frac{1}{2}(q^{\varepsilon_1 N_z} + q^{-\varepsilon_1 N_z})q^{\varepsilon_2 j\frac{N_z}{2}}\bar{\psi} \tag{76}$$

which leads for all j in the limit $q \to 1$ to the Hamiltonian $-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}$ obtained in section 2.2.

The defining relation for the nonlinear term $G_q^j[\bar{\psi}, \psi]$ is given by the last two lines of (74):

$$\begin{aligned} &2\{(G_1\psi_1)(R_2\psi_1 - R_1\psi_2) + (G_2\psi_2)(R_1\psi_2 - R_2\psi_1) + (G_2\psi_1)(R_1\psi_1 + R_2\psi_2) \\ &\quad + (G_1\psi_2)(R_1\psi_1 + R_2\psi_2)\} \equiv G_1h_1(\psi_1, \psi_2, R_1, R_2) + G_2h_2(\psi_1, \psi_2, R_1, R_2) \end{aligned} \tag{77}$$

where

$$
\begin{aligned}
h_1(\psi_1, \psi_2, R_1, R_2) &= 2\psi_1(R_2\psi_1 - R_1\psi_2) + 2\psi_2(R_1\psi_1 + R_2\psi_2) \\
h_2(\psi_1, \psi_2, R_1, R_2) &= 2\psi_2(R_1\psi_2 - R_2\psi_1) + 2\psi_1(R_1\psi_1 + R_2\psi_2).
\end{aligned} \tag{78}
$$

This has to be equated with the nonlinear terms in (68), hence with

$$
C(\varepsilon_1, \varepsilon_2, j) - D([N_z]^2\rho) \tag{79}
$$

where $C(\varepsilon_1, \varepsilon_2, j)$ is given by the third and fourth line in (71), i.e. the nonlinear terms in $B(\varepsilon_1, \varepsilon_2, j)$:

$$
\begin{aligned}
C(\varepsilon_1, \varepsilon_2, j) &= \left\{(-[N_z]q^{\varepsilon_2\frac{j}{2}N_z}\psi_2)\left(\left[j\frac{N_z}{2}\right](q^{\varepsilon_1 N_z} + q^{-\varepsilon_1 N_z})\psi_1\right)\right. \\
&\quad \left. +([N_z]q^{\varepsilon_2\frac{j}{2}N_z}\psi_1)\left(\left[j\frac{N_z}{2}\right](q^{\varepsilon_1 N_z} + q^{-\varepsilon_1 N_z})\psi_2\right)\right\} \\
&= \frac{1}{2\mathrm{i}}\left\{(-[N_z]q^{\varepsilon_2\frac{j}{2}N_z}\psi)\left(\left[j\frac{N_z}{2}\right](q^{\varepsilon_1 N_z} + q^{-\varepsilon_1 N_z})\bar{\psi}\right)\right. \\
&\quad \left. +([N_z]q^{\varepsilon_2\frac{j}{2}N_z}\bar{\psi})\left(\left[j\frac{N_z}{2}\right](q^{\varepsilon_1 N_z} + q^{-\varepsilon_1 N_z})\psi\right)\right\}.
\end{aligned} \tag{80}
$$

Dependent on the choice of the shift R (*q-assumption*), different values for G_1 and G_2 can be derived from the last line in (77) together with (79). G_1 and G_2 so obtained lead to the nonlinear term in the q-Schrödinger equation (73) via the relation (compare with the notation above (74))

$$
F_{NL} \equiv (G_q^j[\bar{\psi}, \psi]\psi)(R\bar{\psi}) = ((G_1 + \mathrm{i}G_2)\psi)(R_1 + \mathrm{i}R_2)\bar{\psi}. \tag{81}
$$

We emphasize that the nonlinear term F_{NL} depends on the choices for R_1 and R_2.

A calculation shows that in the limit $q \to 1$, the nonlinearity is independent on $j \in 2\mathbb{N}$. One always obtains the same imaginary part for the nonlinear functional, i.e. (28) which was also derived without q-deformation. In addition, three types of real parts occur. With the notation $R = R_1 + \mathrm{i}R_2 = (aq^{\alpha N_z} + bq^{\beta N_z}) + \mathrm{i}(cq^{\gamma N_z} + dq^{\delta N_z})$ (recall also that $\psi = \psi_1 + i\psi_2$), these limits are (for different $a, b, c, d, \alpha, \beta, \gamma, \delta$) given by

(i)

$$
A\frac{\psi''\bar{\psi}' - \bar{\psi}''\psi'}{\psi\bar{\psi}' - \bar{\psi}\psi'} \tag{82}
$$

with $A = \frac{\varepsilon_2 j(a+b)}{8(a\alpha+b\beta)}$ if $R_2 = 0$ and $A = \frac{-\varepsilon_2 j(c+d)}{8(c\gamma+d\delta)}$ if $R_1 = 0$. If $R = S_1$, $A = \frac{1}{2}$ holds.

(ii)

$$
B\frac{D\rho''}{2\rho} \tag{83}
$$

i.e. equal to the limit of the imaginary part, with $B = \pm\frac{a+b}{c+d}$. In the special case $R_1 = R_2 = 1$, $B = \pm 1$ respectively.

(iii) For $R = 1$ (trivial shift operator) the real part remains undetermined by (74) as in the undeformed case.

Finally, we give the result for $\theta \neq 0$:

$$
\begin{aligned}
(\mathrm{i}\partial_t\psi)\bar{\psi} &= \left([N_z]\left[j\frac{N_z}{2}\right][j]^{-1}\psi\right)(S_1\bar{\psi}) - \mathrm{i}\left[j\frac{\theta}{2}\right][j]^{-1}\left\{q^{-\varepsilon_2\frac{j}{2}\theta}(\mathrm{i}[N_z]q^{-\varepsilon_2\frac{j}{2}N_z}\psi)\right. \\
&\quad \left. +q^{\varepsilon_2\frac{j}{2}\theta}(\mathrm{i}[N_z]q^{\varepsilon_2\frac{j}{2}N_z}\psi)q^{-\varepsilon_2 jN_z}\right\}(S_1\bar{\psi}) + F_{NL}
\end{aligned} \tag{84}
$$

with $S_1 = \frac{1}{2}(q^{\varepsilon_1 N_z} + q^{-\varepsilon_1 N_z})q^{\varepsilon_2 j\frac{N_z}{2}}$. It is discussed in section 4.4.

4.3. q-Deformation of the time-derivative

So far, we have considered the usual time-derivative ∂_t. However, it is also possible to replace it by a q-time derivative. This could be useful in the following sense. If we choose $R = S = S_1$ in (73) (which means that the real part of the nonlinearity is given by (82)), then it is reasonable to ask whether one can construct a $\mathrm{D}_{q,t}$ which compensates this shift. The occurrence of a space shift in $\mathrm{D}_{q,t}$ is in principle possible and sometimes even necessary [7]. It relates the t and ϕ-dependence of $\psi(\phi, t)$ and leads to a restriction of the evolution equation. This q-assumption corresponds to the following ansatz for a q-Schrödinger equation

$$\begin{aligned} (\mathrm{i}D_{q,t}\psi)(T\circ Y\bar{\psi}) &= (H_q^j\psi)(S\bar{\psi}) + (G_q^j\psi)(R\bar{\psi}) \\ -(\mathrm{i}D_{q,t}\bar{\psi})(\bar{T}\circ\bar{Y}\psi) &= (\bar{H}_q^j\bar{\psi})(\bar{S}\psi) + (\bar{G}_q^j\bar{\psi})(R\psi) \end{aligned} \tag{85}$$

where Y, S and R are shifts in space and T and $\bar{T}$ in time. One can choose $Y = R = S = S_1 = \frac{1}{2}(q^{\varepsilon_1 N_z} + q^{-\varepsilon_1 N_z})q^{\varepsilon_2 j\frac{N_z}{2}}$ (compare with (75)). $\mathrm{D}_{q,t}$ is constructed by a 'shifted Leibniz rule' of the form

$$\mathrm{D}_{q,t}(\bar{\psi}\psi) = (\mathrm{D}_{q,t}\,\bar{\psi})(\bar{T}\circ\bar{Y}\psi) + (\mathrm{D}_{q,t}\,\psi)(T\circ Y\bar{\psi}) \tag{86}$$

and is given by

$$\mathrm{D}_{q,t} = S_1\frac{\bar{T}f(z) - Tf(z)}{q - q^{-1}} \tag{87}$$

i.e. it is defined by the time shifts. With (86) and (66) we obtain the same Schrödinger equation in the limit $q \to 1$ as before, although the q-assumptions are different.

4.4. Discussion of the q-Schrödinger equation

The set of q-Schrödinger equations (84) consists of nonlinear difference equations which contain the parameter j. For $q \to 1$, some of the nonlinearities vanish, but the D-dependence remains also in the continuous case. It is remarkable that for $q \to 1$ for all choices for R and S the same nonlinear imaginary part, given by (28), is obtained.

Since j occurs in the evolution equations in the combination $[\frac{j}{2}N_z]$, its interpretation is the following: whereas difference operators $[N_z]$ usually operate on nearest-neighbour points, also difference operators involving next nearest ($j = 4$) and further points come into play, here. Thus the parameter j, which was introduced as a mathematical necessity in (34) in order to close the q-Witt algebra, indicates the range of the interaction between points in S_N^1. In particular, the range of the interaction blows up with j.

It is instructive to write the set of q-Schrödinger equations (84) directly as a difference equation in dependence of j. We find

$$\begin{aligned} (\mathrm{i}\partial_t\psi(l))\bar{\psi}(l) = {} & \frac{1}{2(q-q^{-1})}\cdot\frac{1}{(q^j - q^{-j})} \\ & \times\left(\psi\left(l+1+\frac{j}{2}\right) - \psi\left(l+1-\frac{j}{2}\right) - \psi\left(l-1+\frac{j}{2}\right)\right. \\ & \left.+\psi\left(l-1-\frac{j}{2}\right)\right)\left(\bar{\psi}\left(l+1+\frac{\varepsilon_2 j}{2}\right) + \bar{\psi}\left(l-1+\frac{\varepsilon_2 j}{2}\right)\right) \\ & + \left[j\frac{\theta}{2}\right][j]^{-1}q^{-\varepsilon_2\frac{j}{2}\theta}\left(\psi\left(l+1-\frac{\varepsilon_2 j}{2}\right) - \psi\left(l-1-\frac{\varepsilon_2 j}{2}\right)\right) \end{aligned}$$

$$\left(\bar{\psi}\left(l+1+\frac{\varepsilon_2 j}{2}\right)+\bar{\psi}\left(l-1+\frac{\varepsilon_2 j}{2}\right)\right)\frac{1}{2(q-q^{-1})}$$
$$+\left[j\frac{\theta}{2}\right][j]^{-1}q^{\varepsilon_2\frac{j}{2}\theta}\left(\psi\left(l+1+\frac{\varepsilon_2 j}{2}\right)-\psi\left(l-1+\frac{\varepsilon_2 j}{2}\right)\right)$$
$$\left(\bar{\psi}\left(l+1-\frac{\varepsilon_2 j}{2}\right)+\bar{\psi}\left(l-1-\frac{\varepsilon_2 j}{2}\right)\right)\frac{1}{2(q-q^{-1})}+F_{NL}. \tag{88}$$

The Hamiltonian relates via F_{NL} the points $l+2$, l and $l-2$ independently of j. In addition, the points $l+1+\frac{j}{2}$, $l+1-\frac{j}{2}$, $l-1+\frac{j}{2}$ and $l-1-\frac{j}{2}$ are related, i.e. the coefficient j determines the range of 'the interaction' (as shown in figure 1). Via F_{NL} no further dependences are introduced and none are cancelled. To calculate the limit $q\to 1$ one should use the Fourier transform $\psi(l+m)=\sum_{n=0}^{N-1}\psi_{l,n}\exp(\frac{2\pi\mathrm{i}}{N}ln)\exp(\frac{2\pi\mathrm{i}}{N}lm)=\sum_{n=0}^{N-1}\psi_{l,n}z_l^n q^{ln}$ to exhibit the q-dependence.

It is difficult to decide on the range of the interaction, i.e. on j, by physical considerations. Thus, it is suggestive—also in view of the physical meaning of the algebra generated by (49–51)—to assume that all j have to be treated on an equal footing, i.e. a *democratic regime*:

$$H_q=\frac{1}{N}\sum_{j=0}^{N-1}H_q^j. \tag{89}$$

This is consistent with our theory and the nonlinear part G_q^j and P_q^j can be treated similarly.

5. Summary and outlook

The observation that measurements of momentum are related in classical physics to two consecutive positional measurements have led us to the use of difference operators for quantum mechanical momentum observables instead of differential operators. The introduction of particular difference quotients called q-derivatives was realized via a q-deformation of the kinematical algebra on S^1, i.e. the inhomogeneous Witt algebra, and implemented in the framework of Borel quantization of a system moving on the simplest compact configuration space, the circle S^1 and its N-point discretization S_N^1. Some additional assumptions, called q-assumptions, were necessary to get a reasonable q-deformation, similarly as for the nondeformed case some (but different) assumptions are necessary to get a reasonable quantization. The generalized Ehrenfest theorem could be formulated also in the deformed case on S_N^1 and it has been an implicit condition for the evolution equation for ψ, which has led to a set of difference Schrödinger equations parametrized by $j\in\mathbb{N}$, which are highly nonlinear difference equations for the time evolution.

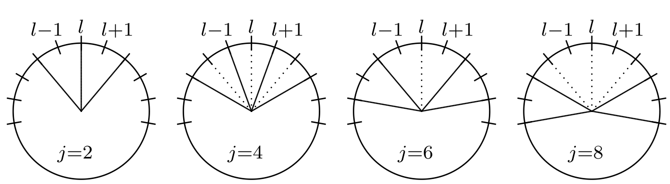

Figure 1. The broken lines give the interaction range for $D=0$. The dotted lines indicate which interactions appear in addition for $D\neq 0$.

This last part is of *particular* interest. For $q \to 1$, the different time evolutions give the usual Schrödinger equation, but with a fixed imaginary *and* real nonlinear term, *depending on the q-assumptions*. Borel quantization on S^1 gives the same result, but with an arbitrary real nonlinear term. So, the q-assumptions, necessary for our procedure, lead for $q \to 1$ to a fixed nonlinear term in the Schrödinger equation, which means that one can learn on the continuous case if it is viewed as a limit of the q-deformed case.

We emphasize that the Schrödinger equation derived via q-deformation already on the level of the quantization of the kinematical algebra as presented here differs qualitatively from those approaches, where discretization takes place via a substitution of differentials by q-derivatives in the Schrödinger equation. The latter would lead in the framework of Borel quantization as presented in section 2 (for $D = 0$) via a replacement of $\frac{\mathrm{d}}{\mathrm{d}\phi}$ by the q-derivative D_q in (3), to a q-Schrödinger equation of the form

$$\mathrm{i}\partial_t \psi(l) = -\tfrac{1}{2}\,\mathrm{D}_q^2\,\psi(l) - \mathrm{i}\theta\,\mathrm{D}_q\,\psi(l) \tag{90}$$

which differs considerably from our result (84).

Finally, we remark that our discussion is valid for a particular configuration space, S^1 and its discretization; it would be interesting to derive a similar result for more general configuration spaces. We leave this issue for a future investigation.

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